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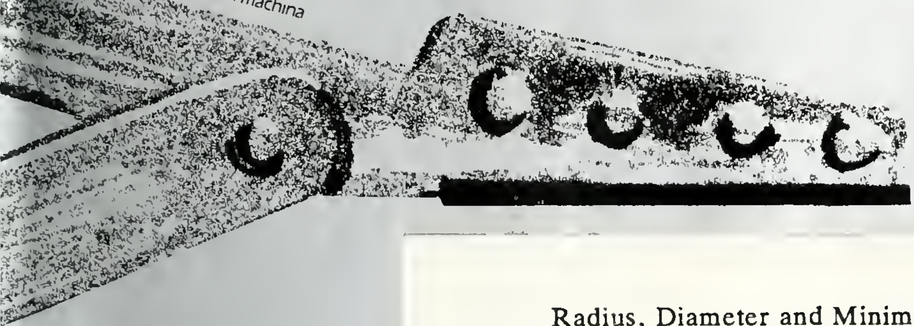

Radius, Diameter and Minimum Degree

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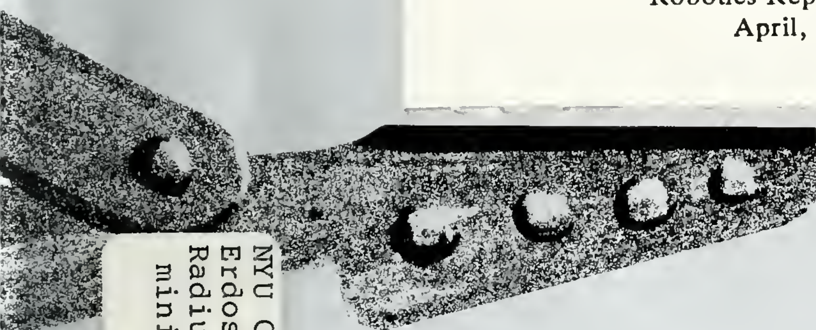

New York University
C Institute of Mathematical Sciences

Computer Science Division
251 Mercer Street New York, N.Y. 10012

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New York University
Dept. of Computer Science
Courant Institute of Mathematical Sciences
251 Mercer Street
New York, New York 10012

Hungarian Academy of Sciences
H-13264 Budapest Pf. 127
Hungary

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Radius, Diameter and Minimum Degree

Paul Erdös, János Pach, Richard Pollack and Zsolt Tuza

Hungarian Academy of Sciences, Budapest
Courant Institute of Mathematical Sciences, New York

Abstract

We give asymptotically sharp upper bounds for the maximum diameter and radius of (i) a connected graph, (ii) a connected triangle-free graph, (iii) a connected C_4-free graph with n vertices and with minimum degree δ, where n tends to infinity. Some conjectures for K_r-free graphs are also stated.

Let G be a connected graph with vertex set V(G) and edge set E(G). For any $x,y \in V(G)$ let $d_G(x,y)$ denote the <u>distance</u> between x and y, i.e., the minimum length of an x-y path in G. The <u>diameter</u> and the <u>radius</u> of G are defined as

$$\operatorname{diam} G = \max_{x,y \in V(G)} d_G(x,y) ,$$

$$\operatorname{rad} G = \min_{x \in V(G)} \max_{y \in V(G)} d_G(x,y) .$$

The following theorem answers a question of Gallai.

Theorem 1. Let G be a connected graph with n vertices and with minimum degree $\delta \geq 2$. Then

(i) $$\operatorname{diam} G \leq \left[\frac{3n}{\delta+1}\right] - 1$$

(ii) $$\operatorname{rad} G \leq \frac{3}{2}\,\frac{n-3}{\delta+1} + 5 .$$

Furthermore, (i) and (ii) are tight apart from the exact value of the additive constants, and for every $\delta > 5$ equality can hold in (i) for infinitely many values of n.

Proof. Let G be a graph of diameter $d > 1$ and minimum degree δ, and assume that it is saturated, i.e., the addition of any edge results in a graph with smaller diameter. Let x and y be two vertices with $d_G(x,y) = d$, and put $S_i = \{v \in V(G): d_G(x,v) = i\}$ for any $0 \leq i \leq d$. Then $|S_0| = |S_d| = 1$ and by the condition on the minimum degree

$$|S_{i-1}| + |S_i| + |S_{i+1}| \geq \delta + 1 \quad \text{for all } 0 \leq i \leq d ,$$

where $S_{-1} = S_{d+1} = \emptyset$. It can readily be checked that if $d > 2$ then this implies

(1) $$n = \sum_{i=0}^{d} |S_i| \geq \left(\left[\frac{d}{3}\right] + 1\right)(\delta + 1) + \varepsilon_d ,$$

where ε_d denotes the remainder of d upon division by 3. This yields (i). Further, it is easily seen that equality can be attained in (1) for any pair $d \geq 2$, $\delta \geq 2$.

Note that (i) is tight e.g. for the following graph. Let $k > 1$, $\delta > 5$ and $V(G) = V_0 \cup V_1 \cup \ldots \cup V_{3k-1}$, where

$$|V_i| = \begin{cases} 1 \ , \text{ if } i \equiv 0 \text{ or } 2 \pmod 3 \ , \\ \delta \ , \text{ if } i = 1 \text{ or } 3k-2 \ , \\ \delta - 1 \ , \text{ otherwise} \ . \end{cases}$$

Let two vertices $v \in V_i$, $v' \in V_j$ be joined by an edge of G if and only if $|j-i| \leq 1$.

To prove (ii), let us fix a <u>center</u> x of G, i.e., a point for which $\max_{y \in V(G)} d_G(x,y) = \text{rad } G = r$, and put $S_i = \{v \in V(G): d_G(x,y) = i\}$ for $0 \leq i \leq r$. Given any $v \in S_i$, pick a point $v' \in S_{i-1}$ such that $vv' \in E(G)$, $(1 \leq i \leq r)$. The collection of the edges $\{vv': v \in V(G) - \{x\}\}$ obviously defines a spanning tree $T \subseteq G$ with the property that

$$d_T(x,y) = d_G(x,y) \quad \text{for all } y \in V(G) \ .$$

Let $T(x,y)$ denote the path connecting x and y in T. Further, put

$$S_{\leq j} = \bigcup_{0 \leq i \leq j} S_i \ , \quad S_{\geq j} = \bigcup_{j \leq i \leq r} S_i \ .$$

Fix a point $y' \in S_r$. A vertex $y'' \in V(G)$ is said to be <u>related to</u> y', if one can find $\bar{y} \in T(x,y') \cap S_{\geq 5}$ and $\bar{y}'' \in T(x,y'') \cap S_{\geq 5}$ such that

(2) $$d_G(\bar{y}', \bar{y}'') \leq 2 \ .$$

There are two cases to consider.

<u>Case A</u>: There exists a point $y'' \in S_{\geq r-5}$ which is not related to y'.

For any i, let S'_i (and S''_i) denote the set of all elements of S_i whose distance from at least one point of $T(x,y') \cap S_{\geq 5}$ (one point of $T(x,y'') \cap S_{\geq 5}$, resp.) is at most 1 in G. Using the fact that y' and y'' are not related ,

$$\left(\bigcup_{i=4}^{r} S_i'\right) \cap \left(\bigcup_{i=4}^{r} S_i''\right) = \emptyset .$$

On the other hand, by the condition on the minimum degree,

$$|S_{i-1}'| + |S_i'| + |S_{i+1}'| \geq \delta + 1 \quad \text{for all } 5 \leq i \leq r ,$$

$$|S_{i-1}''| + |S_i''| + |S_{i+1}''| \geq \delta + 1 \quad \text{for all } 5 \leq i \leq s ,$$

where $s = d_G(x,y'') \geq r-5$. Similarly to (1), we now obtain

$$n \geq |S_{\leq 3}| + \sum_{i=4}^{r} |S_i'| + \sum_{i=4}^{s+1} |S_i''|$$

$$\geq \delta+2+ \{\frac{1}{3}\sum_{i=5}^{r} (|S_{i-1}'| + |S_i'| + |S_{i+1}'|) + 1\} + \{\frac{1}{3}\sum_{i=5}^{s} (|S_{i-1}''| + |S_i''| + |S_{i+1}''|) + 1\}$$

$$\geq \delta+4+\frac{1}{3}(r-4)(\delta+1) + \frac{1}{3}(s-4)(\delta+1) \geq \frac{1}{3}(2r-10)(\delta+1) + 3 ,$$

whence (ii) follows immediately.

Case B: Every point $y'' \in S_{\geq r-5}$ is related to y'

Let x' denote the only element of $T(x,y')$ which belongs to S_5. Then, for any $y \in S_{\leq r-6}$,

$$d_G(x',y) \leq d_G(x',x) + d_G(x,y) \leq 5 + r - 6 = r - 1 .$$

On the other hand, every $y'' \in S_{\geq r-5}$ is related to y', therefore by (2)

$$d_G(x',y'') \leq d_G(x',\bar{y}') + d_G(\bar{y}',\bar{y}'') + d_G(\bar{y}'',y'')$$

$$\leq (d_G(x,\bar{y}') - 5) + 2 + (r - d_G(x,\bar{y}''))$$

$$\leq r - 3 + d_G(\bar{y}',\bar{y}'') \leq r - 1 .$$

Thus, $d_G(x',y) \leq r-1$ for every $y \in V(G)$, contradicting our assumption that rad $G = r$. This completes the proof of (ii). □

Theorem 2. Let G be a connected triangle-free graph with n vertices and with minimum degree $\delta \geq 2$. Then

(i) $$\operatorname{diam} G \leq 4\left\lceil \frac{n-\delta-1}{2\delta} \right\rceil$$

(ii) $$\operatorname{rad} G \leq \frac{n-2}{\delta} + 12 .$$

Furthermore, (i) and (ii) are tight apart from the exact value of the additive constant, and for every $\delta \geq 2$ equality can hold in (i) for infinitely many values of n.

Proof. Let x and y be two vertices of G with $d_G(x,y) = \operatorname{diam} G = d$, and put $S_i = \{v \in V(G): d_G(x,v) = i\}$ for any $0 \leq i \leq d$.

For every i exactly one of the following two possibilities occurs. Either S_i does not contain any edge of G and then

(3) $$|S_{i-1}| + |S_{i+1}| \geq \delta ,$$

or $vv' \in E(G)$ for some $v,v' \in S_i$, and then the neighborhoods of v and v' are disjoint, therefore

(4) $$|S_{i-1}| + |S_i| + |S_{i+1}| \geq 2\delta .$$

Note that (3) and (4) immediately imply that

(5) $$|S_{i-1}| + |S_i| + |S_{i+1}| + |S_{i+2}| \geq 2\delta \quad \text{for every } 0 \leq i \leq d-1 ,$$

where $S_{-1} = S_{d+1} = \emptyset$. Indeed, if S_i or S_{i+1} contains an edge, then (5) follows from (4). Otherwise, by (3), $|S_{i-1}| + |S_{i+1}| \geq \delta$ and $|S_i| + |S_{i+2}| \geq \delta$, hence (5) is true again.

Now easy calculations show that

$$n \geq ([\tfrac{d}{4}] + 1)2\delta - 1 + \begin{cases} -\delta+2 & \text{if } d \equiv 0 \pmod 4, \\ 1 & \text{if } d \equiv 1 \pmod 4, \\ 2 & \text{if } d \equiv 2 \pmod 4, \\ 3 & \text{if } d \equiv 3 \pmod 4, \end{cases}$$

and equality can hold for every pair $d, \delta \geq 2$. This yields (i). Note that (i) is tight e.g. for the following graphs. Let $V(G) = V_0 \cup V_1 \cup \ldots \cup V_{4k}$ with

$$|V_i| = \begin{cases} 1 & \text{if } i \equiv 0 \text{ or } 1 \pmod 4 \text{ and } i \neq 1, \\ \delta & \text{if } i = 1 \text{ or } 4k-1, \\ \delta-1 & \text{otherwise}, \end{cases}$$

and assume that V_i and V_{i+1} induce a complete bipartite subgraph of G for every i.

The proof of the second part of the theorem is vey similar to that of Theorem 1 (ii). We use the same notation and terminology as there, with the following modification. Fix a point $y' \in S_r$. A vertex $y'' \in V(G)$ is now said to be <u>related to</u> y', if there exist $\bar{y}' \in T(x,y') \cap S_{\geq 9}$ and $\bar{y}'' \in T(x,y'') \cap S_{\geq 9}$ such that

(2') $$d_G(\bar{y}', \bar{y}'') \leq 4 .$$

<u>Case A</u>: There exists a point $y'' \in S_{\geq r-9}$ which is not related to y'.

For any i, let S'_i (S''_i) denote the set of all elements of S_i whose distance from at least one point of $T(x,y') \cap S_{\geq 9}$ ($T(x,y'') \cap S_{\geq 9}$, resp.) is at most 2. Then

$$\left(\bigcup_{i=7}^{r} S'_i\right) \cap \left(\bigcup_{i=7}^{r} S''_i\right) = \emptyset ,$$

and by an argument similar to the proof of (5) we obtain

$$|S'_{i-1}| + |S'_i| + |S'_{i+1}| + |S'_{i+2}| \geq 2\delta \quad \text{for all } 8 \leq i \leq r-1 ,$$

$$|S''_{i-1}| + |S''_i| + |S''_{i+1}| + |S''_{i+2}| \geq 2\delta \quad \text{for all } 8 \leq i \leq s-1 ,$$

where $s = d_G(x,y'') \geq r-9$. This yields

$$n \geq |S_{\leq 6}| + \sum_{i=7}^{r} |S'_i| + \sum_{i=7}^{s+1} |S''_i| \geq (r-12)\delta + 2$$

and (ii) follows.

Case B: Every point of $S_{\geq r-9}$ is related to y'.

A slight modification of the argument which settled the corresponding case in Theorem 1 shows that this cannot occur. □

Theorem 3. Let $\delta \geq 2$ be a fixed integer, and let G be a connected, C_4-free graph with n vertices and with minimum degree δ. Then

(i) $$\text{diam } G \leq \frac{5n}{\delta^2 - 2[\frac{\delta}{2}]+1} ,$$

(ii) $$\text{rad } G \leq \frac{5n}{2(\delta^2 - 2[\frac{\delta}{2}]+1)} + O(1) .$$

Furthermore, if δ is large, then these bounds are almost tight. More precisely, if $\delta+1$ is a prime power, then there exists a graph G with the above properties and

(iii) $$\text{diam } G \geq \frac{5n}{\delta^2+3\delta+2} - 1 .$$

Proof. Let $x_0x_1x_2\cdots x_d$ be a chordless path of length $d = \operatorname{diam} G$ in G. Put $S_{\leq 2}(x) = \{v \in V(G): d_G(x,v) \leq 2\}$ for any $x \in V(G)$. Since G does not contain C_4 ,

$$|S_{\leq 2}(x)| \geq \delta^2 - 2[\tfrac{\delta}{2}] + 1 \quad \text{for every } x \in V(G) .$$

In view of the fact that

$$S_{\leq 2}(x_{5i}) \cap S_{\leq 2}(x_{5j}) = \emptyset \quad \text{for all } 0 \leq i \neq j \leq d/5 ,$$

we obtain

$$n \geq ([\tfrac{d}{5}] + 1)(\delta^2 - 2[\tfrac{\delta}{2}] + 1) ,$$

which proves (i). From here (ii) follows in exactly the same way as before.

To establish (iii), set $p = \delta+1$ and let H denote the following graph discovered by Brown [3] and Erdös-Rényi [4]. Let $V(H)$ consist of all ordered triples $\underline{x} = (x_1,x_2,x_3) \neq \underline{0}$ whose elements are taken from $GF(p)$, where two triples $\underline{x}$ and $\underline{x}'$ are considered identical if $\underline{x}' = \lambda\underline{x}$ for some $\lambda \in GF(p)$. Let $\underline{x}\ \underline{y} \in E(H)$ if any only if $\underline{x}\cdot\underline{y} = 0$. Clearly, H is C_4-free, and has $p^2 + p + 1$ vertices, each of degree p or $p+1$.

Let us fix $\underline{u},\underline{v},\underline{z} \in V(H)$ satisfying $\underline{u}\cdot\underline{z} = \underline{v}\cdot\underline{z} = \underline{z}\cdot\underline{z} = 0$. Let $\underline{u}_0 = \underline{z}, \underline{u}_1, \underline{u}_2,\cdots,\underline{u}_p$ and $\underline{v}_0 = \underline{z},\underline{v}_1,\underline{v}_2,\cdots,\underline{v}_p$ denote the neighbors of $\underline{u}$ and $\underline{v}$, respectively. For every i $(1 \leq i \leq p)$ there is a uniquely determined $j(i)$ $(1 \leq j(i) \leq p)$ such that $\underline{u}_i\underline{v}_{j(i)} \in E(H)$. On the other hand, no $\underline{u}_i$ or $\underline{v}_j$ $(1 \leq i,j \leq p)$ is adjacent to $\underline{z}$ in H.

Let H_0 denote the graph obtained from H after the removal of the vertex $\underline{z}$ and all edges of the form $\underline{u}_i\underline{v}_{j(i)}$, $1 \leq i \leq p$. It is clear that $d_{H_0}(\underline{u},\underline{v}) = 4$, and the minimum degree of the vertices of H_0 is $p-1 = \delta$.

Let G be defined as the union of k disjoint isomorphic copies $H_0^{(1)}, H_0^{(2)}, \cdots, H_0^{(k)}$ of H_0, and let us make it connected by adding the edges $\underline{v}^{(t)}\underline{u}^{(t+1)}$ for every $1 \leq t < k$. Then $|V(G)| = n = k(p^2+p) = k(\delta^2+3\delta+2)$ and

$$\text{diam } G = 5k - 1 = \frac{5n}{\delta^2+3\delta+2} - 1 . \qquad \square$$

<u>Conjecture</u>. Let $r,\delta > 1$ be fixed natural numbers, and let G be a connected graph with n vertices and with minimum degree δ.

(i) If G is K_{2r}-free and δ is a multiple of $(r-1)(3r+2)$, then

$$\text{diam } G \leq \frac{2(r-1)(3r+2)}{(2r^2-1)\delta} n + O(1) \quad \text{while } n \to +\infty .$$

(ii) If G is K_{2r+1}-free and δ is a multiple of $3r-1$, then

$$\text{diam } G \leq \frac{3r-1}{r\delta} n + O(1) \quad \text{while } n \to +\infty .$$

These bounds, if valid, are asymptotically sharp, as is shown by the following graphs.

(i): Let $V(G) = \bigcup_{i=0}^{k} \bigcup_{j=1}^{r(i)} V_{ij}$, where $r(i) = r$ or $r-1$

depending on whether i is even or odd, and let

$$|V_{ij}| = \begin{cases} \dfrac{r\delta}{(r-1)(3r+2)} & \text{if } i\neq 0,k \text{ is even} \\ \\ \dfrac{(r+1)\delta}{(r-1)(3r+2)} & \text{if } i\neq 0,k \text{ is odd}\ , \end{cases}$$

and $|V_{0j}| = |V_{kj}| = \delta$ for every j. Let two vertices $v \in V_{ij}$ and $v' \in V_{i'j'}$ be joined by an edge if and only if (a) $|i-i'| = 1$ or (b) $i=i'$ and $j\neq j'$. Then G is obviously K_{2r}-free.

(ii): Let $V(G) = \bigcup_{i=0}^{k} \bigcup_{j=1}^{r} V_{ij}$, where $|V_{ij}| = \frac{\delta}{3r-1}$ if $i\neq 0,k$ and $|V_{0j}| = |V_{kj}| = \delta$ $(1 \leq j \leq r)$. Let the edge set of G be defined by the same rule as above. Then G is K_{2r+1}-free.

For an extensive survey of problems and results on the relations between the degrees, the radius and the diameter of a graph see Chapter 4 in Bollobás [2], or Bermond-Bollobás [1].

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